

# FORMAL GEOMETRIC QUANTISATION FOR PROPER ACTIONS

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ABSTRACT. We define formal geometric quantisation for proper Hamiltonian actions by possibly noncompact groups on possibly noncompact, prequantised symplectic manifolds, generalising work of Weitsman and Paradan. We study the functorial properties of this version formal geometric quantisation, and relate it to a recent result by the authors via a version of the shifting trick.

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## INTRODUCTION

Consider a Hamiltonian action by a compact Lie group  $K$  on a possibly non-compact prequantised symplectic manifold  $(N, \nu)$ , with proper momentum map. Weitsman [19] defined the *formal geometric quantisation* of this action, which by definition commutes with reduction:

$$Q_K^{-\infty}(N, \nu) = \sum_{\lambda \in \Lambda_+^K} Q(N_\lambda, \nu_\lambda)[\pi_\lambda^K],$$

where  $\Lambda_+^K$  is the set of dominant integral weights of  $K$ , with respect to a maximal torus and positive root system, and  $\pi_\lambda^K$  is the irreducible representation of  $K$  with highest weight  $\lambda \in \Lambda_+^K$ . For such  $\lambda$ , the compact symplectic manifold or orbifold

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2010 *Mathematics Subject Classification*. Primary 53D50, Secondary 19K56, 22D25, 19L47.

*Key words and phrases*. Formal geometric quantization, locally compact groups, momentum map, Hamiltonian manifold, Hochs–Landsman conjecture, Guillemin–Sternberg conjecture.

The first author was supported by the European Union, through a Marie Curie fellowship. The second author acknowledges funding by the Australian Research Council, through Discovery Projects DP110100072 and DP130103924.

$(N_\lambda, \nu_\lambda)$  is the symplectic reduction of the given action at  $\lambda/i$ . Formal geometric quantisation takes values in the *generalised representation ring*

$$R^{-\infty}(K) = \text{Hom}_{\mathbb{Z}}(R(K), \mathbb{Z}),$$

where  $R(K)$  is the usual representation ring of  $K$ .

Paradan [16] proved that formal geometric quantisation is functorial with respect to Cartesian products and restriction to subgroups. These two properties imply that it is compatible with the ring structure on  $R^{-\infty}(K)$ . Ma and Zhang ([12] and [13]), and also Paradan [17] proved that quantisation commutes with reduction, in the sense that

$$(0.1) \quad Q_K(N, \nu) = Q_K^{-\infty}(N, \nu),$$

for a certain definition of the quantisation  $Q_K(N, \nu) \in R^{-\infty}(K)$ .

On the other hand, Landsman [10] proposed a definition of geometric quantisation of a Hamiltonian action by a Lie group  $G$  on a prequantised symplectic manifold  $(M, \omega)$ , if the orbit space  $M/G$  is compact. He used the *analytic assembly map* from the Baum–Connes conjecture [1], which takes values in the  $K$ -theory group  $K_*(C_r^*(G))$  of the (full or reduced)  $C^*$ -algebra of the group  $G$ . Applying this assembly map to a Dirac operator coupled to a prequantum line bundle yields Landsman's definition of

$$Q_G(M, \omega) \in K_*(C_r^*(G)).$$

Mathai and Zhang [14] showed that Landsman's version of quantisation commutes with reduction at the trivial representation, at least if one multiplies the symplectic form  $\omega$  by a large enough integer. For (possibly only presymplectic) manifolds of the form  $M = G \times_K N$ , with  $N$  a prequantised Hamiltonian  $K$ -manifold, it was shown in [6] that

$$Q_G(M, \omega) = \sum_{\lambda \in \Lambda_+^K} Q(M_{\lambda+\rho_c}, \omega_{\lambda+\rho_c})[\lambda].$$

Here  $[\lambda]$  is a certain generator of  $K_*(C_r^*(G))$ ,  $\rho_c$  is half the sum of the compact positive roots, and  $(M_{\lambda+\rho_c}, \omega_{\lambda+\rho_c})$  is the symplectic reduction of the action at  $(\lambda + \rho_c)/i$ . The shift over  $\rho_c$  appears because  $\text{Spin}^c$ -quantisation is used rather than Dolbeault-quantisation.

A common generalisation of  $R^{-\infty}(K)$  and  $K_*(C_r^*(G))$  is the  $K$ -homology group  $K^*(C_r^*(G))$  of  $C_r^*(G)$ . In view of the successes for quantisation with values in  $R^{-\infty}(K)$  and  $K_*(C_r^*(G))$ , it makes sense to find a definition of quantisation with values in  $K^*(C_r^*(G))$ , without assuming the group or the orbit space to be compact. In this note, we generalise the formal quantisation studied by Weitsman and Paradan to noncompact groups. We study its functorial properties, and give a relation with the main result in [8] via a version of the shifting trick.

## 1. COMPACT GROUPS

Let  $K$  be a compact, connected Lie group, with Lie algebra  $\mathfrak{k}$ . Let  $\Lambda_+^K$  be the set of dominant integral weights of  $K$ , with respect to a maximal torus and a choice of positive roots. For  $\lambda \in \Lambda_+^K$ , let  $\pi_\lambda^K$  be the irreducible representation of  $K$  with highest weight  $\lambda$ . The *generalised representation ring* of  $K$  is defined as

$$R^{-\infty}(K) = \text{Hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$$

of  $K$ . Here  $R(K)$  denotes the usual representation ring. The generalised representation ring is generated by the elements  $[\pi_\lambda^K]^*$ , for  $\lambda \in \Lambda_+^K$ , where

$$[\pi_\lambda^K]^*([\pi_{\lambda'}^K]) = \delta_{\lambda\lambda'} := \begin{cases} 1 & \text{if } \lambda = \lambda'; \\ 0 & \text{if } \lambda \neq \lambda', \end{cases}$$

for all  $\lambda' \in \Lambda_+^K$ .

Let  $(N, \nu)$  be a prequantised symplectic manifold, equipped with a Hamiltonian  $K$ -action. Suppose the momentum map  $\Phi^K : N \rightarrow \mathfrak{k}^*$  is proper. Then for every  $\lambda \in \Lambda_+^K$ , the symplectic reduction [11]  $(N_\lambda, \nu_\lambda)$  of the action at  $\lambda/i$  is compact. Hence it has a quantisation  $Q(N_\lambda, \nu_\lambda) \in \mathbb{Z}$ , where one can use Meinrenken and Sjamaar's approach [15] in the singular case.

Weitsman [19] introduced the *formal geometric quantisation*  $Q_K^{-\infty}(N, \nu)$  of the action by  $K$  on  $(N, \nu)$ , as

$$Q_K^{-\infty}(N, \nu) = \sum_{\lambda \in \Lambda_+^K} Q(N_\lambda, \nu_\lambda) [\pi_\lambda^K]^* \in R^{-\infty}(K).$$

Paradan [16] proved that formal quantisation is functorial with respect to restriction to subgroups, and also notes that it is functorial with respect to Cartesian products. Explicitly, let  $K' < K$  be a closed subgroup, with Lie algebra  $\mathfrak{k}'$ . Suppose that the momentum map  $\Phi^{K'} : N \rightarrow (\mathfrak{k}')^*$  for the action by  $K'$  on  $N$  is still proper. Then Paradan showed that

$$(1.1) \quad \text{Res}_{K'}^K(Q_K^{-\infty}(N, \nu)) = Q_{K'}^{-\infty}(N, \nu).$$

Here  $\text{Res}_{K'}^K : R^{-\infty}(K) \rightarrow R^{-\infty}(K')$  is the restriction map.

In addition, for  $j = 1, 2$ , let  $K_j$  be a compact, connected Lie group, and let  $(N_j, \nu_j)$  be a prequantised Hamiltonian  $K_j$ -manifold with proper momentum map. Then Paradan points out that

$$(1.2) \quad Q_{K_1}^{-\infty}(N_1, \nu_1) \otimes Q_{K_2}^{-\infty}(N_2, \nu_2) = Q_{K_1 \times K_2}^{-\infty}(N_1 \times N_2, \nu_1 \times \nu_2).$$

The properties (1.1) and (1.2) together imply that, if  $K_1 = K_2 = K$ ,

$$(1.3) \quad Q_K^{-\infty}(N_1, \nu_1) \times Q_K^{-\infty}(N_2, \nu_2) = Q_K^{-\infty}(N_1 \times N_2, \nu_1 \times \nu_2),$$

with respect to the product  $\times$  in the ring  $R^{-\infty}(K)$ .

Our goal is to generalise the definition of formal geometric quantisation, and its functoriality properties with respect to restriction and Cartesian products, to noncompact groups. The generalised representation ring will be replaced by  $K$ -homology of group  $C^*$ -algebras.

## 2. $K$ -HOMOLOGY OF GROUP $C^*$ -ALGEBRAS

Let  $G$  be a connected Lie group containing  $K$  as a maximal compact subgroup. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Let  $C_r^*G$  be the reduced  $C^*$ -algebra of  $G$ . We will write  $d := \dim(G/K)$ . The *Connes–Kasparov* conjecture, proved for almost connected groups by Chabert, Echterhoff and Nest [2], states that there is an isomorphism of Abelian groups

$$\text{D-Ind}_K^G : R(K) \xrightarrow{\cong} K_d(C_r^*G).$$

This isomorphism is called *Dirac induction*, and is given by

$$\text{D-Ind}_K^G[\pi_\lambda^K] = \mu_{G/K}^G[D_{G/K}^\lambda],$$

for  $\lambda \in \Lambda_+^K$ , where  $\mu_{G/K}^G$  is the analytic assembly map [1], and  $D_{G/K}^\lambda$  is a Dirac operator on  $G/K$  coupled to the representation  $\pi_\lambda^K$ .

Let  $K^d(C_r^*G)$  be the  $K$ -homology group of  $C_r^*G$  in degree  $d$ . Since  $K_d(C_r^*G) \cong R(K)$  is torsion-free, the universal coefficient theorem [18] states that

$$(2.1) \quad K^d(C_r^*G) \cong \text{Hom}_{\mathbb{Z}}(K_d(C_r^*G), \mathbb{Z}).$$

(In particular,  $R^{-\infty}(K) = K^0(C_r^*K)$ .) The isomorphism is given by the Kasparov product. Therefore, pulling back along the Dirac induction map defines an isomorphism of Abelian groups

$$(2.2) \quad (\text{D-Ind}_K^G)^* : K^d(C_r^*G) \xrightarrow{\cong} R^{-\infty}(K).$$

For  $\lambda \in \Lambda_+^K$ , we write  $[\lambda]$  for the generator  $\text{D-Ind}_K^G[\pi_\lambda^K]$  of  $K^d(C_r^*G)$ . Let  $[\lambda]^* \in K^d(C_r^*G)$  be the corresponding generator, defined by

$$[\lambda]^*([\lambda']) = \delta_{\lambda\lambda'},$$

for  $\lambda' \in \Lambda_+^K$ . Then

$$(2.3) \quad (\text{D-Ind}_K^G)^*[\lambda]^* = [\pi_\lambda^K]^*.$$

We consider  $K_d(C_r^*G)$  as a subgroup of  $K^d(C_r^*G)$  via the map  $[\lambda] \mapsto [\lambda]^*$ , i.e. via the Kasparov product.

### 3. FORMAL GEOMETRIC QUANTISATION

Let  $(M, \omega)$  be a prequantised symplectic manifold, equipped with a proper Hamiltonian  $G$ -action. Suppose the momentum map  $\Phi^G : M \rightarrow \mathfrak{g}^*$  is  $G$ -proper, in the sense that the inverse image of every cocompact set is cocompact. (By a cocompact set we mean a set with compact quotient by the group action.) Then all symplectic reductions of the action are compact.

**Definition 3.1.** The *formal geometric quantisation* of the action by  $G$  on  $(M, \omega)$  is

$$Q_G^{-\infty}(M, \omega) = \sum_{\lambda \in \Lambda_+^K} Q(M_\lambda, \omega_\lambda) [\lambda]^* \in K^d(C_r^*G).$$

Now suppose that  $G$  is semisimple, and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition. Suppose  $M$  is of the form  $M = G \times_K N$  considered in [5] and [6], the quotient of  $G \times N$  by the  $K$ -action

$$k \cdot (g, n) = (gk^{-1}, kn),$$

for  $k \in K$ ,  $g \in G$  and  $n \in N$ . As in [6], consider the  $G$ -invariant presymplectic form (i.e. closed two-form)  $\omega$  on  $M$  given by

$$\omega_{[e, n]}(Tq(X + v), Tq(Y + w)) := \nu_n(v, w) - \langle \Phi^K(n), [X, Y] \rangle,$$

where  $n \in N$ ,  $X, Y \in \mathfrak{p}$ ,  $v, w \in T_n N$ , and  $q : G \times N \rightarrow M$  is the quotient map. If  $G$  has discrete series representations and  $\Phi^K(M)$  lies inside the set of strongly elliptic elements (the ones with compact stabilisers under the coadjoint action), then  $\omega$  is an actual symplectic form (see [5], Proposition 2.4, with more details given in Proposition 12.4 in [4]). In general,  $\omega$  may be degenerate, but all constructions relevant to quantisation and reduction still apply. In particular, for  $\lambda \in \Lambda_+^K$ , one has a (pre)symplectomorphism

$$(3.1) \quad (M_\lambda, \omega_\lambda) \cong (N_\lambda, \nu_\lambda)$$

(see [6], Lemma 5.1), so that  $(M_\lambda, \omega_\lambda)$  is actually a symplectic manifold. It is compact, which also follows from the fact that the momentum map

$$\Phi^G[g, n] = \text{Ad}^*(g)\Phi^K(n),$$

for  $g \in G$  and  $n \in N$ , is  $G$ -proper if  $\Phi^K$  is proper.

**Lemma 3.2.** *In this setting, one has*

$$(\text{D-Ind}_K^G)^*(Q_G^{-\infty}(M, \omega)) = Q_K^{-\infty}(N, \nu).$$

*Proof.* This follows from (2.3) and (3.1).  $\square$

Lemma 3.2 is a formal version of the *quantisation commutes with induction* principle in [5] and [6]. For formal quantisation, we see that one gets this principle almost for free.

*Remark 3.3.* Because of Lemma 3.2, the equality (0.1) implies that

$$Q_G^{-\infty}(M, \omega) = \left( (\text{D-Ind}_K^G)^* \right)^{-1} Q_K^{-\infty}(N, \nu) = \left( (\text{D-Ind}_K^G)^* \right)^{-1} Q_K(N, \nu).$$

Hence, for manifolds of the form  $M = G \times_K N$ , any definition of

$$Q_G(M, \omega) \in K^d(C_r^*G)$$

such that

$$(\text{D-Ind}_K^G)^*(Q_G(M, \omega)) = Q_K(N, \nu)$$

commutes with reduction, in the sense that

$$Q_G(M, \omega) = Q_G^{-\infty}(M, \omega).$$

For compact  $N$ , it was shown in [6] that Landsman's definition of quantisation has this property.

#### 4. A RESTRICTION MAP

We return to the case where  $G$  is any connected Lie group. Let  $G' < G$  be a closed, connected subgroup that has a maximal compact subgroup  $K'$  contained in  $K$ . We write  $d' := \dim(G'/K')$ .

**Definition 4.1.** The *Dirac restriction* map  $\text{D-Res}_{G'}^G$  is defined by commutativity of the following diagram:

$$\begin{array}{ccc} K^d(C_r^*G) & \xrightarrow{\text{D-Res}_{G'}^G} & K^{d'}(C_r^*G') \\ \text{(D-Ind}_K^G)^* \downarrow \cong & & \cong \downarrow \text{(D-Ind}_{K'}^{G'})^* \\ R^{-\infty}(K) & \xrightarrow{\text{Res}_{K'}^K} & R^{-\infty}(K'). \end{array}$$

Because the Dirac restriction map is modelled on the restriction map from  $K$  to  $K'$ , it may not contain all representation theoretic information concerning restriction of representations from  $G$  to  $G'$ . It does have natural functoriality properties with respect to formal quantisation, as we will see.

**Example 4.2.** Suppose  $G$  is semisimple with discrete series. Then  $d$  is even. Let  $\rho_c$  be half the sum of the positive roots of  $K$ . Let  $\lambda \in \Lambda_+^K$ , and suppose  $\lambda + \rho_c$  is strongly elliptic. Let  $\pi_\lambda^G$  be the irreducible discrete series representation of  $G$  with Harish–Chandra parameter  $\lambda + \rho_c$ . Then  $\pi_\lambda^G$  defines a  $K$ -theory class

$$[\pi_\lambda^G] \in K_0(C_r^*G)$$

(see [9], Section 2.2). In (5.3) in [6], it is noted that

$$[\pi_\lambda^G] = (-1)^{d/2}[\lambda] = (-1)^{d/2} \text{D-Ind}_K^G[\pi_\lambda^K].$$

Hence the image of  $[\pi_\lambda^G]$  in  $K^d(C_r^*G)$  is  $[\pi_\lambda^G]^* := (-1)^{d/2}[\lambda]^* \in K^0(C_r^*G)$ , and

$$(\text{D-Ind}_K^G)^*([\pi_\lambda^G]^*) = (-1)^{d/2}[\pi_\lambda^K]^*.$$

Let  $\Lambda_+^{K'}$  be the set of dominant integer weights of  $K'$  with respect to a maximal torus and positive roots compatible with the choices made for  $K$ . Write

$$\text{Res}_{K'}^K(\pi_\lambda^K) = \sum_{\lambda' \in \Lambda_+^{K'}} m_{\lambda'} \pi_{\lambda'}^{K'},$$

for certain integer coefficients  $m_{\lambda'}$ . Then

$$\begin{aligned} (\text{D-Ind}_{K'}^{G'})^* \circ \text{D-Res}_{G'}^G[\pi_\lambda^G]^* &= \text{Res}_{K'}^K \circ (\text{D-Ind}_K^G)^*[\pi_\lambda^G]^* \\ &= (-1)^{d/2} \sum_{\lambda' \in \Lambda_+^{K'}} m_{\lambda'} [\pi_{\lambda'}^{K'}]^*. \end{aligned}$$

Hence

$$\text{D-Res}_{G'}^G[\pi_\lambda^G]^* = (-1)^{d/2} \sum_{\lambda' \in \Lambda_+^{K'}} m_{\lambda'} [\lambda']^*,$$

by (2.3).

Suppose that  $G$  and  $G'$  are semisimple, and that  $M$  is of the form  $M = G \times_K N$  as above. Then formal geometric quantisation has the following functoriality property with respect to Dirac restriction.

**Proposition 4.3.** *One has*

$$\text{D-Res}_{G'}^G(Q_G^{-\infty}(G \times_K N)) = Q_{G'}^{-\infty}(G' \times_{K'} N).$$

*Proof.* Applying Lemma 3.2 and Paradan’s result (1.1), we find that

$$\begin{aligned} (\text{D-Ind}_{K'}^{G'})^* \circ \text{D-Res}_{G'}^G(Q_G^{-\infty}(G \times_K N)) &= \text{Res}_{K'}^K \circ (\text{D-Ind}_K^G)^* Q_G^{-\infty}(G \times_K N) \\ &= \text{Res}_{K'}^K Q_K^{-\infty}(N) \\ &= Q_{K'}^{-\infty}(N) \\ &= (\text{D-Ind}_{K'}^{G'})^*(Q_{G'}^{-\infty}(G' \times_{K'} N)). \end{aligned}$$

□

## 5. PRODUCTS OF GENERATORS

In Section 5.3 of [5], a multiplicativity property of the analytic assembly map is discussed. This will allow us to generalise the multiplicative property (1.2) of formal geometric quantisation to noncompact groups.

Let  $G_1$  and  $G_2$  be locally compact groups, acting properly and cocompactly on locally compact Hausdorff spaces  $X_1$  and  $X_2$ , respectively. There are Kasparov product maps on equivariant  $K$ -homology and on  $K$ -theory,

$$\begin{aligned} K_*^{G_1}(X_1) \times K_*^{G_2}(X_2) &\xrightarrow{\times} K_*^{G_1 \times G_2}(X_1 \times X_2); \\ K_*(C_r^*G_1) \times K_*(C_r^*G_2) &\xrightarrow{\times} K_*(C_r^*(G_1 \times G_2)). \end{aligned}$$

By Theorem 5.2 in [5], the assembly maps  $\mu_{X_j}^{G_j}$  and  $\mu_{X_1 \times X_2}^{G_1 \times G_2}$  satisfy

$$(5.1) \quad \mu_{X_1}^{G_1}(a_1) \times \mu_{X_2}^{G_2}(a_2) = \mu_{X_1 \times X_2}^{G_1 \times G_2}(a_1 \times a_2),$$

for all  $a_j \in K_*^{G_j}(X_j)$ , at least if  $X_1$  and  $X_2$  are metrisable.

Now suppose  $G_1$  and  $G_2$  are connected, semisimple Lie groups. Let  $K_j < G_j$  be maximal compact subgroups, and suppose that  $G_j/K_j$  is spin for  $j = 1, 2$ . (This is always true for certain covers of the groups  $G_j$ .) Write  $d_j := \dim(G_j/K_j)$ .

**Lemma 5.1.** *Let  $\lambda_j \in \Lambda_+^{K_j}$ . Then one has*

$$[\lambda_1] \times [\lambda_2] = [(\lambda_1, \lambda_2)] \in K_{d_1+d_2}(C_r^*(G_1 \times G_2)).$$

(Note that  $\Lambda_+^{K_1 \times K_2} = \Lambda_+^{K_1} \times \Lambda_+^{K_2}$ .)

*Proof.* Let  $G$  be any connected, semisimple Lie group, with a maximal compact subgroup  $K < G$  such that  $G/K$  is spin. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition, and let  $\{X_1, \dots, X_n\}$  be a basis of  $\mathfrak{p}$ , orthonormal with respect to an  $\text{Ad}(K)$ -invariant inner product. Let  $\Delta_{\mathfrak{p}}$  be the standard representation of  $\text{Spin}(\mathfrak{p})$ , and let  $c : \mathfrak{p} \rightarrow \text{End}(\Delta_{\mathfrak{p}})$  be the Clifford action. Let  $\lambda \in \Lambda_+^K$ , and let  $V_\lambda$  be the representation space of  $\pi_\lambda^K$ . Then

$$\text{D-Ind}_K^G[\pi_\lambda^K] = \mu_{G/K}^G[D_{G/K}^\lambda],$$

where  $D_{G/K}^\lambda$  is the Dirac operator

$$D_{G/K}^\lambda = \sum_{j=1}^n X_j \otimes c(X_j) \otimes 1_{V_\lambda}$$

on

$$(C^\infty(G) \otimes \Delta_{\mathfrak{p}} \otimes V_\lambda)^K.$$

Let  $K_j < G_j$  be as above. In  $K_*^{G_1 \times G_2}((G_1 \times G_2)/(K_1 \times K_2))$ , one has for all  $\lambda_j \in \Lambda_+^{K_j}$ ,

$$\begin{aligned} [D_{(G_1 \times G_2)/(K_1 \times K_2)}^{(\lambda_1, \lambda_2)}] &= [D_{G_1/K_1}^{\lambda_1} \otimes 1 + 1 \otimes D_{G_2/K_2}^{\lambda_2}] \\ &= [D_{G_1/K_1}^{\lambda_1}] \times [D_{G_2/K_2}^{\lambda_2}] \end{aligned}$$

Here we have used the fact that  $\pi_{(\lambda_1, \lambda_2)}^{K_1 \times K_2} = \pi_{\lambda_1}^{K_1} \otimes \pi_{\lambda_2}^{K_2}$ . We conclude that, because of (5.1),

$$\begin{aligned} [(\lambda_1, \lambda_2)] &= \text{D-Ind}_{K_1 \times K_2}^{G_1 \times G_2} [\pi_{(\lambda_1, \lambda_2)}^{K_1 \times K_2}] \\ &= \mu_{(G_1 \times G_2)/(K_1 \times K_2)}^{G_1 \times G_2} [D_{(G_1 \times G_2)/(K_1 \times K_2)}^{(\lambda_1, \lambda_2)}] \\ &= \mu_{(G_1 \times G_2)/(K_1 \times K_2)}^{G_1 \times G_2} \left( [D_{G_1/K_1}^{\lambda_1}] \times [D_{G_2/K_2}^{\lambda_2}] \right) \\ &= \mu_{G_1/K_1}^{G_1} [D_{G_1/K_1}^{\lambda_1}] \times \mu_{G_2/K_2}^{G_2} [D_{G_2/K_2}^{\lambda_2}] \\ &= [\lambda_1] \times [\lambda_2]. \end{aligned}$$

□

We will use an extension of Lemma 5.1 to an equality involving the Kasparov product map on  $K$ -homology

$$(5.2) \quad K^*(C_r^*G_1) \times K^*(C_r^*G_2) \xrightarrow{\times} K^*(C_r^*(G_1 \times G_2)).$$

**Corollary 5.2.** *For all  $\lambda_j \in \Lambda_+^{K_j}$ , one has*

$$[\lambda_1]^* \times [\lambda_2]^* = [(\lambda_1, \lambda_2)]^* \in K^{d_1+d_2}(C_r^*(G_1 \times G_2)).$$

*Proof.* Let  $\lambda_j, \mu_j \in \Lambda_+^{K_j}$ . Then

$$(5.3) \quad [(\lambda_1, \lambda_2)]^*([(\mu_1, \mu_2)]) = \delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} = [\lambda_1]^*([\mu_1]) \cdot [\lambda_2]^*([\mu_2]).$$

The isomorphism (2.1) is induced by the Kasparov product, i.e. for  $\lambda \in \Lambda_+^K$ , the homomorphism

$$[\lambda]^* : KK(\mathbb{C}, C_r^*G) \rightarrow \mathbb{Z}$$

is given by taking the Kasparov product with  $[\lambda]^* \in KK(C_r^*G, \mathbb{C})$ . Hence the right hand side of (5.3) equals

$$([\mu_1] \times_{C_r^*G_1} [\lambda_1]^*) \cdot ([\mu_2] \times_{C_r^*G_2} [\lambda_2]^*) = ([\mu_1] \times [\mu_2]) \times_{C_r^*G_1 \otimes C_r^*G_2} ([\lambda_1]^* \times [\lambda_2]^*),$$

where we have used the associativity properties of the Kasparov product. By Lemma 5.1, the latter expression equals

$$([\mu_1, \mu_2]) \times_{C_r^*G_1 \otimes C_r^*G_2} ([\lambda_1]^* \times [\lambda_2]^*) = ([\lambda_1]^* \times [\lambda_2]^*)([(\mu_1, \mu_2)]).$$

□

## 6. MULTIPLICATIVITY

Corollary 5.2 implies that formal quantisation is multiplicative. For  $j = 1, 2$ , let  $(M_j, \omega_j)$  be equivariantly prequantised proper Hamiltonian  $G_j$ -manifolds, with  $G_j$ -proper momentum maps. Suppose the groups  $G_j$  are connected and semisimple.

**Corollary 6.1.** *One has*

$$Q_{G_1 \times G_2}^{-\infty}(M_1 \times M_2, \omega_1 \times \omega_2) = Q_{G_1}^{-\infty}(M_1, \omega_1) \times Q_{G_2}^{-\infty}(M_2, \omega_2) \in K^{d_1+d_2}(C_r^*(G_1 \times G_2)).$$

*Proof.* Let  $\lambda_j \in \Lambda_+^{K_j}$ . As noted by Paradan [16], one has an equality of symplectic reductions

$$((M_1 \times M_2)_{(\lambda_1, \lambda_2)}, (\omega_1 \times \omega_2)_{(\lambda_1, \lambda_2)}) \cong ((M_1)_{\lambda_1} \times (M_2)_{\lambda_2}, (\omega_1)_{\lambda_1} \times (\omega_2)_{\lambda_2}).$$

Since the manifolds  $(M_j)_{\lambda_j}$  are compact, one has

$$Q((M_1)_{\lambda_1}, (\omega_1)_{\lambda_1}) Q((M_2)_{\lambda_2}, (\omega_2)_{\lambda_2}) = Q((M_1)_{\lambda_1} \times (M_2)_{\lambda_2}, (\omega_1)_{\lambda_1} \times (\omega_2)_{\lambda_2}) \in \mathbb{Z}.$$



Hence, because  $\Lambda_+^{K_1 \times K_2} = \Lambda_+^{K_1} \times \Lambda_+^{K_2} := \Lambda_+$ , Corollary 5.2 implies that

$$\begin{aligned} Q_{G_1 \times G_2}^{-\infty}(M_1 \times M_2, \omega_1 \times \omega_2) &= \sum_{(\lambda_1, \lambda_2) \in \Lambda_+} Q((M_1 \times M_2)_{(\lambda_1, \lambda_2)}, (\omega_1 \times \omega_2)_{(\lambda_1, \lambda_2)}) \\ &= \sum_{(\lambda_1, \lambda_2) \in \Lambda_+} Q((M_1)_{\lambda_1}, (\omega_1)_{\lambda_1}) Q((M_2)_{\lambda_2}, (\omega_2)_{\lambda_2}) [\lambda_1]^* \times [\lambda_2]^* \\ &= Q_{G_1}^{-\infty}(M_1, \omega_1) \times Q_{G_2}^{-\infty}(M_2, \omega_2). \end{aligned}$$

□

The compatibility property (1.3) of formal quantisation with the ring structure on  $R^{-\infty}(K)$  can be generalised to noncompact groups. It is possible to equip  $K^d(C_r^*G)$  with a ring structure (rather than just viewing it as an Abelian group), via the isomorphism (2.2).

**Lemma 6.2.** *The product of two elements  $a, b \in K^d(C_r^*G)$  is explicitly given by*

$$ab = \text{D-Res}_{\Delta(G)}^{G \times G}(a \times b),$$

where  $\Delta(G) < G \times G$  is the diagonal subgroup, and  $\times$  denotes the Kasparov product (5.2).

*Proof.* It is enough to check the equality for generators  $a = [\lambda_1]^*$  and  $b = [\lambda_2]^*$ , for  $\lambda_j \in \Lambda_+^K$ . Then, using (2.3) and Corollary 5.2, one finds that

$$\begin{aligned} (\text{D-Ind}_K^G)^*([\lambda_1]^*[\lambda_2]^*) &= \left( (\text{D-Ind}_K^G)^*([\lambda_1]^*) \right) \left( (\text{D-Ind}_K^G)^*([\lambda_2]^*) \right) \\ &= [\pi_{\lambda_1}^K]^* [\pi_{\lambda_2}^K]^* \\ &= \text{Res}_K^{K \times K} [\pi_{\lambda_1}^K \otimes \pi_{\lambda_2}^K]^* \\ &= \text{Res}_K^{K \times K} [\pi_{(\lambda_1, \lambda_2)}^{K \times K}]^* \\ &= \text{Res}_K^{K \times K} \circ (\text{D-Ind}_{K \times K}^{G \times G})^* [(\lambda_1, \lambda_2)]^* \\ &= (\text{D-Ind}_K^G)^* \circ \text{D-Res}_G^{G \times G} [(\lambda_1, \lambda_2)]^* \\ &= (\text{D-Ind}_K^G)^* \circ \text{D-Res}_G^{G \times G} ([\lambda_1]^* \times [\lambda_2]^*). \end{aligned}$$

□

It follows from Proposition 4.3, Corollary 6.1 and Lemma 6.2 that for all pre-quantised Hamiltonian  $K$ -manifolds  $(N_j, \nu_j)$  with proper momentum maps,

$$Q_G^{-\infty}(G \times_K N_1) Q_G^{-\infty}(G \times_K N_2) = Q_G^{-\infty}(G \times_K (N_1 \times N_2)),$$

where  $K$  acts diagonally on  $N_1 \times N_2$ .

## 7. THE SHIFTING TRICK

As in [8], consider a  $G$ -invariant metric on the trivial bundle  $M \times \mathfrak{g}^* \rightarrow M$ , equipped with the  $G$ -action

$$g \cdot (m, \xi) = (g \cdot m, \text{Ad}^*(g)\xi),$$

for  $g \in G$ ,  $m \in M$  and  $\xi \in \mathfrak{g}^*$ . Denote the induced norm on the fibre at  $m$  by  $\|\cdot\|_m$ . Let  $\mathcal{H}$  be the associated norm-squared function of the momentum map  $\Phi^G$ :

$$\mathcal{H}(m) = \|\Phi^G(m)\|_m^2.$$

Consider the one-form  $d_1\mathcal{H} \in \Omega^1(M)$  defined by

$$(d_1\mathcal{H})_m = d_m(m' \mapsto \|\Phi^G(m')\|_m^2).$$

Let  $\text{Crit}_1(\mathcal{H})$  be the set of zeroes of  $d_1\mathcal{H}$ . Under the assumptions that  $\text{Crit}_1(\mathcal{H})/G$  is compact and  $G$  is unimodular, the invariant quantisation

$$Q(M, \omega)^G \in \mathbb{Z}$$

was defined in [8]. It was proved that for  $p \in \mathbb{N}$  large enough,

$$(7.1) \quad Q(M, p\omega)^G = Q(M_0, p\omega_0),$$

and conjectured that this equality holds for  $p = 1$ .

Let  $\lambda \in \Lambda_+^K$ . Consider the reduction map  $R_\lambda^G : K^d(C_r^*G) \rightarrow \mathbb{Z}$  given by taking the multiplicity of  $[\lambda]^*$ . Let  $\mathcal{O}_\lambda^- := \text{Ad}^*(G)\lambda/i$  be the coadjoint orbit through  $\lambda/i$ , equipped with minus the standard Kirillov–Kostant–Souriau symplectic form. Let  $\omega^\lambda \in \Omega^2(M \times \mathcal{O}_\lambda^-)$  be the induced product symplectic form. Let  $\mathcal{H}_\lambda$  be the function  $\mathcal{H}$  defined above, for the diagonal action by  $G$  on  $M \times \mathcal{O}_\lambda^-$ . If the conjecture that (7.1) holds for  $p = 1$  is true, then one has the following version of the shifting trick.

**Proposition 7.1.** *If  $\text{Crit}_1(\mathcal{H}_\lambda)/G$  is compact, then, if  $\lambda/i$  is a regular value of  $\Phi^G$ ,*

$$R_\lambda^G(Q_G^{-\infty}(M, \omega)) = Q(M \times \mathcal{O}_\lambda^-, \omega^\lambda)^G.$$

*Proof.* See Corollary 5.12 in [8]. □

Under the stronger assumption (which may be restrictive) that  $\text{Crit}_1(\mathcal{H}_\lambda)/G$  is compact and  $\lambda/i$  is a regular value of  $\Phi^G$  for *all*  $\lambda$ , one can define a *semi-formal* version of quantisation as

$$Q_G^{\text{semi}}(M, \omega) = \sum_{\lambda \in \Lambda_+^K} Q(M \times \mathcal{O}_\lambda^-, \omega^\lambda)^G [\lambda]^*.$$

Then Proposition 7.1 implies that  $Q_G^{\text{semi}}(M, \omega) = Q_G^{-\infty}(M, \omega)$ .

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